Implement Blind Interference Alignment over Homogeneous 3-user 2×1 Broadcast Channel

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Abstract

This paper first studies the homogeneous 3-user 2×1 broadcast channel (BC) with no CSIT. We show a sufficient condition for it to achieve the optimal $\frac{3}{2}$ degrees of freedom (DoF) by using Blind Interference Alignment (BIA). BIA refers to the interference alignment method without the need of CSIT. It further studies the 2×1 broadcast network in which there are $K \geq 3$ homogeneous single-antenna users, and their coherence time offsets are independently and uniformly distributed. We show that, if $K \geq 11$, the two-antenna transmitter can find, with more than 95% certainty, three users to form a BIA-feasible 3-user BC and achieve the optimal $\frac{3}{2}$ DoF.

Index Terms

Blind interference alignment, DoF, homogeneous fading channel, MISO BC.

I. INTRODUCTION

In interference channels with generic channel states, the implementation of Interference Alignment (IA) requires the instantaneous and global channel state information at transmitters (CSIT), such as in the seminal K-user interference channel [1]. However, in practical transmission systems, the delay introduced by achieving CSIT via feedbacks from the receivers makes the instantaneous and global CSIT unrealistic. Technically speaking, the degree-of-freedom (DoF) region achieved by IA with the assumption of perfect CSIT only serves as an upper bound on achievable DoF.

Be more realistic, IA with delayed CSIT [2] and IA with no CSIT [3] are of more significance from engineering point of view. Usually reduced DoF is traded for relaxed requirement on CSIT. Consider a K-user $L \times 1$ broadcast channel (BC) in which a L-antenna transmitter delivers independent data streams to K single-antenna users. When $K = L = 2$, 2 DoF can be achieved by beamforming if CSIT is available, while only $\frac{4}{3}$ DoF is achievable if only delayed CSIT is available. Surprisingly, the DoF of $\frac{4}{3}$ is also achievable by using IA with no CSIT if the BC contains 4 channel uses and has certain structure on its channel state matrix. By contrast, only 1 DoF is achievable for the BC if one has no CSIT and uses no IA.

In this paper, we concentrate on IA with no CSIT, which is frequently referred to as Blind Interference Alignment (BIA) [3], [4], [5]. Prior works in the literature only study what the limit of achievable DoF is by using BIA. For instance, it is shown that the optimal DoF of $\frac{LK}{L+K-1}$ is achievable for a K-user $L \times 1$ BC if it contains finite channel uses and the channel state matrix has certain structure [3], [5]. Whether the optimal DoF is achievable for a general K-user $L \times 1$ BC containing infinite channel uses, and how to achieve the optimal DoF are not well studied except our recent work on homogeneous 2-user 2×1 BC [6].

We further examine the condition for a homogeneous 3-user 2×1 BC to be BIA-feasible, more precisely, to achieve the optimal DoF $\frac{3}{2}$ by using BIA. Unlike homogeneous 2-user BC, which can be characterized by a single block offset between the two users [6], a homogeneous 3-user BC needs at least two block offsets to parameterize. Due to the doubling of parameter number, it becomes tedious and formidable to analyze the 3-user BC by using the method in [6]. In particular, the method deals with the explicit decomposition of a $4 \times 5N$ channel matrix with N being the coherence time. To reduce the complexity, we propose a new analysis method, which simplifies the decomposition of the $4 \times 5N$ channel matrix into the decomposition of a 12×12 matrix regardless of the value of N. By using this new method, we identify a sufficient condition on block offsets for a homogeneous 3-user BC to be BIA-feasible. If there are $K \geq 11$ homogeneous users in a 2×1 broadcast network, we further show that it is highly possible that the two-antenna transmitter can find, from the K users, three users to form a BIA-feasible 3-user BC.

II. SYSTEM MODEL

A 3-user 2×1 broadcast channel (BC) consists of a two-antenna transmitter and three single-antenna users. In this paper, we consider a 3-user 2×1 BC in which the three users undertake independent block fading with the same coherence time, referred to as homogeneous 3-user 2×1 BC. As shown in Fig. 1, let h_{ij} denote the channel coefficient of the link connecting transmit antenna Tx_j, $j \in \{1, 2\}$, to user Rx_i, $i \in \{1, 2, 3\}$. Further define coefficient vector $H_i(n) \triangleq [h_{i1}(n), h_{i2}(n)]^T$ for

$$H_1(n) = \begin{bmatrix} h_{11}(n) & h_{12}(n) \end{bmatrix}^T$$

$$H_2(n) = \begin{bmatrix} h_{21}(n) & h_{22}(n) \end{bmatrix}^T$$

$$H_3(n) = \begin{bmatrix} h_{31}(n) & h_{32}(n) \end{bmatrix}^T$$

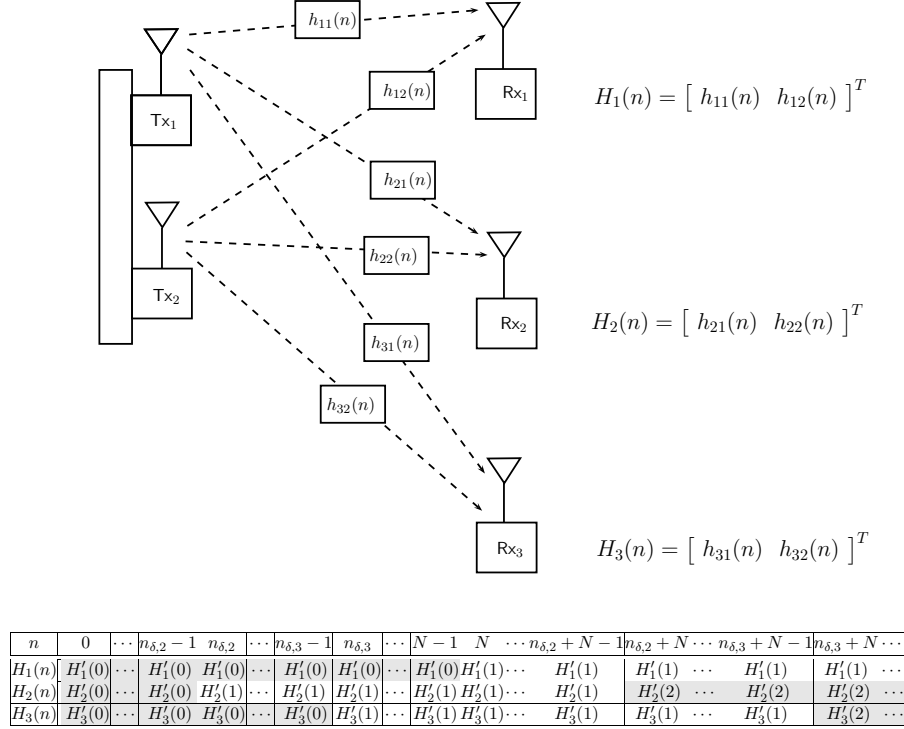| n	0	$\cdots$	$n_{\delta,2}-1$	$n_{\delta,2}$	$\cdots$	$n_{\delta,3}-1$	$n_{\delta,3}$	$\cdots$	$N-1$	N	$\cdots$	$n_{\delta,2}+N-1$	$n_{\delta,2}+N$	$\cdots$	$n_{\delta,3}+N-1$	$n_{\delta,3}+N$	$\cdots$
$H_1(n)$	$H_1'(0)$	$\cdots$	$H_1'(0)$	$H_1'(0)$	$\cdots$	$H_1'(0)$	$H_1'(0)$	$\cdots$	$H_1'(0)$	$H_1'(1)$	$\cdots$	$H_1'(1)$	$H_1'(1)$	$\cdots$	$H_1'(1)$	$H_1'(1)$	$\cdots$
$H_2(n)$	$H_2'(0)$	$\cdots$	$H_2'(0)$	$H_2'(1)$	$\cdots$	$H_2'(1)$	$H_2'(1)$	$\cdots$	$H_2'(1)$	$H_2'(1)$	$\cdots$	$H_2'(1)$	$H_2'(2)$	$\cdots$	$H_2'(2)$	$H_2'(2)$	$\cdots$
$H_3(n)$	$H_3'(0)$	$\cdots$	$H_3'(0)$	$H_3'(0)$	$\cdots$	$H_3'(0)$	$H_3'(1)$	$\cdots$	$H_3'(1)$	$H_3'(1)$	$\cdots$	$H_3'(1)$	$H_3'(1)$	$\cdots$	$H_3'(1)$	$H_3'(2)$	$\cdots$

Fig. 1: System model of a 3-user 2×1 BC.

Rx_i, where $n \geq 0$ is the discrete-time index. Represent the coherence time for $H_i(n)$ by N, which is the same for all users, and the initial time offset for $H_i(n)$ by $n_{\delta,i}$. Without loss of generality, let $n_{\delta,1} = 0$, and $0 \leq n_{\delta,i} < N$ for $i \in \{2,3\}$. So, as illustrated in Fig. 1, we can let $H_i'(a_i)$ denote the coefficient vector for user Rx_i at its a_ith coherence block. In specific, for user Rx_1 we have $H_1(a_1 N + b_1) = H_1'(a_1)$ for all $a_1 \geq 0$ and $0 \leq b_1 < N$; for users Rx_2 and Rx_3, we have $H_i(b_i) = H_i'(0)$ for $0 \leq b_i \leq n_{\delta,i} - 1$, and $H_i(a_i N + n_{\delta,i} + b_i) = H_i'(a_i + 1)$ for $a_i \geq 0$ and $0 \leq b_i < N$, where $i \in \{2,3\}$.

A. Interference alignment

It is known that the optimal achievable DoF, also known as multiplexing gain, is $\frac{3}{2}$ for the 3-user 2×1 BC with no CSIT [4]. As an illustration, Fig. 2 shows how interference alignment helps to achieve the optimal DoF. In this figure, we consider four arbitrary time slots $n_1 < n_2 < n_3 < n_4$, which are not necessarily consecutive. Their channel coefficients are given by

$H_1(n_1)$	$H_1(n_2)$	$H_1(n_3)$	$H_1(n_4)$
$H_2(n_1)$	$H_2(n_2)$	$H_2(n_3)$	$H_2(n_4)$
$H_3(n_1)$	$H_3(n_2)$	$H_3(n_3)$	$H_3(n_4)$

.

Since each transmit antenna delivers one symbol on each time slot, the four-time-slot channel block is also referred to as a *4-symbol channel block*. Over the block, we denote channel coefficient matrix from transmit antenna Tx_j to user Rx_i as $H_{ij} = \mathrm{diag}[h_{ij}(n_1), h_{ij}(n_2), h_{ij}(n_3), h_{ij}(n_4)]$. Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathcal{C}^{4\times 1}$ are the signaling vectors for Tx_1; $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathcal{C}^{4\times 1}$ are the signaling vectors for Tx_2. At user Rx_i, the received signal vector $\mathbf{y}_i \in \mathcal{C}^{4\times 1}$ is

$$\mathbf{y}_i = H_{i1}[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]\mathbf{s}_1 + H_{i2}[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]\mathbf{s}_2 + \mathbf{z}_j \tag{1}$$

where $\mathbf{s}_j = [s_{j1}, s_{j2}, s_{j3}]^T \in \mathcal{C}^{3\times 1}$ represents three independent data streams from transmit antenna Tx_j, $\mathbf{z}_i \in \mathcal{C}^{4\times 1}$ is the AWGN vector at user Rx_i.

To achieve the optimal DoF of $\frac{4}{3}$, the interference alignment implementation shown in Fig. 2 requires

$$\begin{cases} H_{11}\mathbf{v}_2 \rightarrow H_{12}\mathbf{u}_2 \\ H_{11}\mathbf{v}_3 \rightarrow H_{12}\mathbf{u}_3 \end{cases} \tag{2}$$

and

$$\begin{cases} H_{21}\mathbf{v}_1 \rightarrow H_{22}\mathbf{u}_1 \\ H_{21}\mathbf{v}_3 \rightarrow H_{22}\mathbf{u}_3 \end{cases} \tag{3}$$
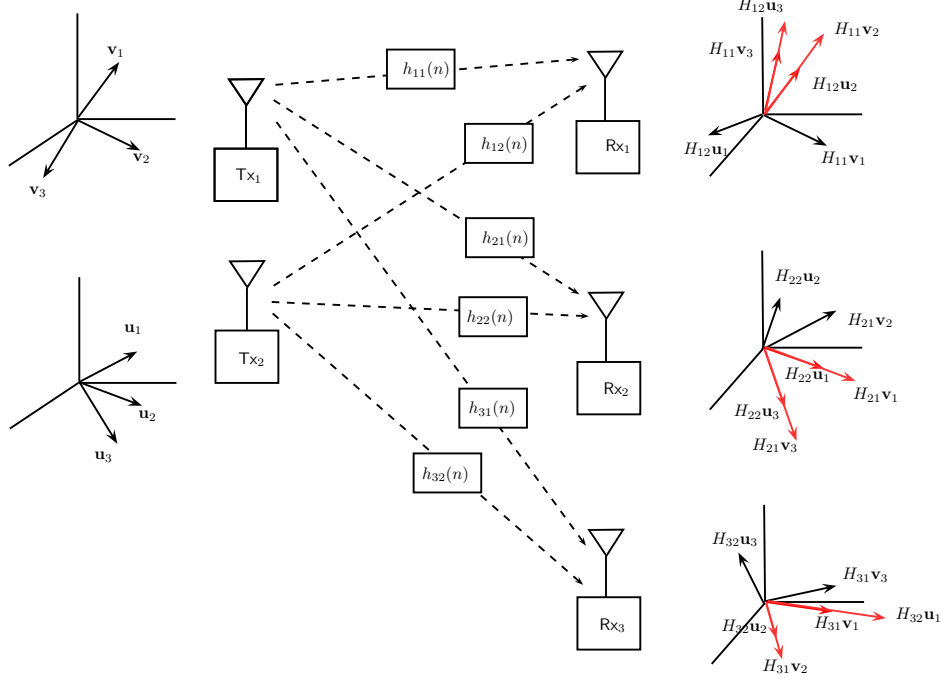
Fig. 2: Interference alignment of a 3-user 2×1 BC. Note that the symbol streams s_{ij} are omitted for simplicity.

$$\begin{cases} \mathbf{v}_1 & \to \mathrm{diag}\left(\frac{h_{22}(n_1)}{h_{21}(n_1)}, \frac{h_{22}(n_2)}{h_{21}(n_2)}, \frac{h_{22}(n_3)}{h_{21}(n_3)}, \frac{h_{22}(n_4)}{h_{21}(n_4)}\right) \mathbf{u}_1 \to \mathrm{diag}\left(\frac{h_{32}(n_1)}{h_{31}(n_1)}, \frac{h_{32}(n_2)}{h_{31}(n_2)}, \frac{h_{32}(n_3)}{h_{31}(n_3)}, \frac{h_{32}(n_4)}{h_{31}(n_4)}\right) \mathbf{u}_1 \\ \mathbf{v}_2 & \to \mathrm{diag}\left(\frac{h_{12}(n_1)}{h_{11}(n_1)}, \frac{h_{12}(n_2)}{h_{11}(n_2)}, \frac{h_{12}(n_3)}{h_{11}(n_3)}, \frac{h_{12}(n_4)}{h_{11}(n_4)}\right) \mathbf{u}_2 \to \mathrm{diag}\left(\frac{h_{32}(n_1)}{h_{31}(n_1)}, \frac{h_{32}(n_2)}{h_{31}(n_2)}, \frac{h_{32}(n_3)}{h_{31}(n_3)}, \frac{h_{32}(n_4)}{h_{31}(n_4)}\right) \mathbf{u}_2 \\ \mathbf{v}_3 & \to \mathrm{diag}\left(\frac{h_{12}(n_1)}{h_{11}(n_1)}, \frac{h_{12}(n_2)}{h_{11}(n_2)}, \frac{h_{12}(n_3)}{h_{11}(n_3)}, \frac{h_{12}(n_4)}{h_{11}(n_4)}\right) \mathbf{u}_3 \to \mathrm{diag}\left(\frac{h_{22}(n_1)}{h_{21}(n_1)}, \frac{h_{22}(n_2)}{h_{21}(n_2)}, \frac{h_{22}(n_3)}{h_{21}(n_3)}, \frac{h_{22}(n_4)}{h_{21}(n_4)}\right) \mathbf{u}_3 \end{cases} . \tag{5}$$

and

$$\begin{cases} H_{31}\mathbf{v}_1 \to H_{32}\mathbf{u}_1 \\ H_{31}\mathbf{v}_2 \to H_{32}\mathbf{u}_2 \end{cases} \tag{4}$$

where $\mathbf{x} \to \mathbf{y}$ means that $\mathbf{x}$ aligns with $\mathbf{y}$, that is, $\mathbf{x} = c\mathbf{y}$ for a nonzero scalar c. In this implementation, Rx_i, $i \in \{1, 2, 3\}$, decodes the symbols delivered by $\mathbf{v}_i$ and $\mathbf{u}_i$, i.e., s_{1i} and s_{2i} from Tx_1 and Tx_2, respectively. In total, six symbols are delivered by four channel uses, so the DoF $\frac{6}{4} = \frac{3}{2}$ is achieved. The alignment criteria given above can be rewritten as (5).

B. BIA-feasible channel block

In this part, we investigate the way to meet the interference alignment condition (5) when we have no CSIT but the knowledge of the coherence time N and the offsets $n_{\delta,i}$, $i \in \{1, 2, 3\}$ at the transmitter. This type of interference alignment requiring no CSIT is referred to as BIA [4].

According to the results in [4], when a 4-symbol channel block presents the following channel coefficient pattern

$H_1'(\beta)$	$H_1'(\alpha)$	$H_1'(\beta)$	$H_1'(\beta)$
$H_2'(\gamma)$	$H_2'(\gamma)$	$H_2'(\psi)$	$H_2'(\gamma)$
$H_3'(\rho)$	$H_3'(\rho)$	$H_3'(\rho)$	$H_3'(\pi)$

,

it is feasible for BIA to achieve the optimal $\frac{3}{2}$. For simplicity of presentation, we say the block is *BIA-feasible*. In any homogeneous 3-user 2×1 BC, one cannot find four time slots to form a BIA-feasible block with the channel pattern above. Fortunately, the channel pattern does not form the necessary condition for a 4-symbol channel block to be BIA-feasible, other channel patterns can also form BIA-feasible 4-symbol channel blocks. For example, a 4-symbol channel block is BIA-feasible if it has the following channel pattern:

$H_1'(\alpha)$	$H_1'(\alpha)$	$H_1'(\beta)$	$H_1'(\beta)$
$H_2'(\gamma)$	$H_2'(\psi)$	$H_2'(\psi)$	$H_2'(\psi)$
$H_3'(\rho)$	$H_3'(\rho)$	$H_3'(\rho)$	$H_3'(\pi)$

$$\begin{cases} \mathbf{v}_1 & \to \mathrm{diag}\left(\frac{h'_{22}(\gamma)}{h'_{21}(\gamma)}, \frac{h'_{22}(\psi)}{h'_{21}(\psi)}, \frac{h'_{22}(\psi)}{h'_{21}(\psi)}, \frac{h'_{22}(\psi)}{h'_{21}(\psi)}\right) \mathbf{u}_1 \to \mathrm{diag}\left(\frac{h'_{32}(\rho)}{h'_{31}(\rho)}, \frac{h'_{32}(\rho)}{h'_{31}(\rho)}, \frac{h'_{32}(\rho)}{h'_{31}(\rho)}, \frac{h'_{32}(\pi)}{h'_{31}(\pi)}\right) \mathbf{u}_1 \\ \mathbf{v}_2 & \to \mathrm{diag}\left(\frac{h'_{12}(\alpha)}{h'_{11}(\alpha)}, \frac{h'_{12}(\alpha)}{h'_{11}(\alpha)}, \frac{h'_{12}(\beta)}{h'_{11}(\beta)}, \frac{h'_{12}(\beta)}{h'_{11}(\beta)}\right) \mathbf{u}_2 \to \mathrm{diag}\left(\frac{h'_{32}(\rho)}{h'_{31}(\rho)}, \frac{h'_{32}(\rho)}{h'_{31}(\rho)}, \frac{h'_{32}(\rho)}{h'_{31}(\rho)}, \frac{h'_{32}(\pi)}{h'_{31}(\pi)}\right) \mathbf{u}_2 \\ \mathbf{v}_3 & \to \mathrm{diag}\left(\frac{h'_{12}(\alpha)}{h'_{11}(\alpha)}, \frac{h'_{12}(\alpha)}{h'_{11}(\alpha)}, \frac{h'_{12}(\beta)}{h'_{11}(\beta)}, \frac{h'_{12}(\beta)}{h'_{11}(\beta)}\right) \mathbf{u}_3 \to \mathrm{diag}\left(\frac{h'_{22}(\gamma)}{h'_{21}(\gamma)}, \frac{h'_{22}(\psi)}{h'_{21}(\psi)}, \frac{h'_{22}(\psi)}{h'_{21}(\psi)}, \frac{h'_{22}(\psi)}{h'_{21}(\psi)}\right) \mathbf{u}_3 \end{cases} . \tag{6}$$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
$H_1(n)$	□	□	□	□	□	□	□	•	•	•	•	•	•	•	△	△	△	△	△	△	△	★	★	★	★	★	★	★	○	○	○	○	○	○
$H_2(n)$	♠	♠	▽	▽	▽	▽	▽	▽	▽	♦	♦	♦	♦	♦	♦	♦	⊖	⊖	⊖	⊖	⊖	⊖	⊖	►	►	►	►	►	►	►	⊗	⊗	⊗	⊗
$H_3(n)$	♣	♣	♣	♣	♡	♡	♡	♡	♡	♡	♡	◄	◄	◄	◄	◄	◄	◄	◇	◇	◇	◇	◇	◇	◇	▼	▼	▼	▼	▼	▼	▼	⊡	⊡
π pattern	π_0	π_0	π_1	π_1	π_2	π_2	π_2	π_3	π_3	π_4	π_4	π_5	π_5	π_5	π_6	π_6	π_7	π_7	π_8	π_8	π_8	π_9	π_9	π_{10}	π_{10}	π_{11}	π_{11}	π_{11}	π_0	π_0	π_1	π_1	π_2	π_2

Fig. 3: A homogeneous BC with $N = 7$, $n_{\delta,2} = 2$ and $n_{\delta,3} = 4$.

To see its BIA-feasibility, we substitute the channel state into (5) and get the alignment conditions given by (6). The conditions would be satisfied if we choose $\mathbf{v}_1 = \mathbf{u}_1 = [0, 1, 1, 0]^T$, $\mathbf{v}_2 = \mathbf{u}_2 = [1, 1, 0, 0]^T$ and $\mathbf{v}_3 = \mathbf{u}_3 = [0, 0, 1, 1]^T$, and thus the channel block achieves interference alignment with no CSIT, i.e., BIA-feasibility.

Over the exemplary 4-symbol channel block shown above, the first two symbols of user Rx_1 have the same channel state while the last two symbols have also the same channel state but different from the first two. To reflect the distribution of these two channel states over time, we say user Rx_1 has the channel structure $(2, 2)$ over the 4-symbol channel block. Similarly, user Rx_2 has the channel structure $(1, 3)$, and user Rx_3 has the channel structure $(3, 1)$. We say the 4-symbol channel block has the channel pattern $\{(2, 2), (1, 3), (3, 1)\}$. It is easy to verify that a 4-symbol channel block is BIA-feasible if its channel pattern is any permutation of $(1, 3)$, $(2, 2)$ and $(3, 1)$, such as $\{(1, 3), (3, 1), (2, 2)\}$. The number of such permutations is six, therefore, one can find six kinds of BIA-feasible 4-symbol channel block from homogeneous 3-user 2×1 BCs.

We say a homogeneous 3-user 2×1 BC with coherence time N and time offsets $n_{\delta,i}$ is *BIA-feasible if the broadcast channel can be fully decomposed into BIA-feasible 4-symbol channel blocks*. We give a simple demonstration to show this argument. Consider a homogeneous BC with $N = 7$, $n_{\delta,2} = 2$ and $n_{\delta,3} = 4$; its channel coefficients over time are presented in Fig. 3, in which the same marks represent the same channel state. The channel block from $n = 6$ to $n = 33$, which contains $4N = 28$ consecutive time slots, can be decomposed into $N = 7$ BIA-feasible 4-symbol channel blocks as presented in Fig. 4. For instance, the time slots $n_1 = 8$, $n_2 = 10$, $n_3 = 12$ and $n_4 = 14$ form a BIA-feasible channel block with the channel pattern $\{(3, 1), (1, 3), (2, 2)\}$. Since the channel repeats the same pattern every 28 consecutive symbols afterwards, the same decomposition can be achieved repetitively. Therefore, this channel can obtain the optimal $\frac{3}{2}$ DoF by using BIA. Note that the first three time slots are negligible or can be utilized for control messages.

III. SUFFICIENT CONDITION FOR BIA-FEASIBILITY

In the example above, we note that, from $n = n_1$ to the next time $n = n_1 + 1$, channel state $\{H_1(n), H_2(n), H_3(n)\}$ may or may not remain unchanged. For instance, the channel state remains the same from $n = 0$ to $n = 1$ in Fig. 3, but varies from $n = 1$ to $n = 2$. We collect consecutive time slots having identical channel states into a group, and assign the time slots in the same group with the same label (π). In the example, the 0th group contains $n = 0, 1$, and is given the label π_0; afterwards, the tth group is given the label π_t until $n = 4N - 1 = 27$th time slot has been considered. For every consecutive block of $4N$ time slots, we repeat the grouping process, and start labeling groups from 0 again.

Next we generalize the grouping and labeling idea to an arbitrary integer N. Without loss of generality, we assume $n_{\delta,2} \le n_{\delta,3}$. Let s_t be the number of π_t's in the tth group. Then we can easily verify that $s_0 = n_{\delta,2}$, $s_1 = n_{\delta,3} - n_{\delta,2}$, $s_2 = N - n_{\delta,3}$ and

$$s_t = s_{t'}, \text{ if } t \equiv t' \mod 3, \tag{7}$$

where $t' \in \{0, 1, 2\}$. The constant term '3' in the equation above reflects the number of users. So, the example given by Fig. 3 has $s_0 = 2$, $s_1 = 2$ and $s_2 = 3$. By this definition, the channel block from $n = 0$ to $n = N - 1$ can be divided into 3 groups, which have sizes s_0, s_1 and s_2, respectively. The "super" channel block from $n = 0$ to $n = 4N - 1$, a block containing four coherence blocks, can be divided into $3 \times 4 = 12$ groups, which have sizes s_t, $t \in \{0, 1, \cdots, 11\}$.

It is clear that a homogeneous 3-user 2×1 BC can be fully characterized by s_t. Based on s_t, we can visualize the BC by constructing a *pattern diagram*. In specific, we align, along the tth column of the pattern diagram, the π_t's in the tth group. As an example, Fig. 5 shows the pattern diagram of the homogeneous BC given by Fig. 3.

6	7	9	11		8	10	12	14		13	15	16	18		17	19	21	23		20	22	24	25		26	28	30	32		27	29	31	33
□	•	•	•		•	•	•	△		△	△	△	△		△	★	★	★		△	★	★	★		★	○	○	○		★	○	○	○
▽	▽	♦	♦		▽	♦	♦	♦		♦	♦	⊖	⊖		⊖	⊖	⊖	►		⊖	⊖	►	►		►	►	⊗	⊗		►	►	⊗	⊗
♡	♡	♡	◄		♡	♡	◄	◄		◄	◄	◄	◇		◄	◇	◇	◇		◇	◇	◇	▼		▼	▼	▼	⊡		▼	▼	▼	⊡
π_2	π_3	π_4	π_5		π_3	π_4	π_5	π_6		π_5	π_6	π_7	π_8		π_7	π_8	π_9	π_{10}		π_8	π_9	π_{10}	π_{11}		π_{11}	π_0	π_1	π_2		π_{11}	π_0	π_1	π_2

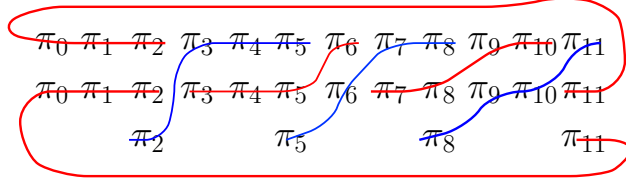Fig. 4: The implementation of BIA for the homogeneous BC shown in Fig. 3.

Fig. 5: The pattern diagram for the homogeneous BC with $N = 7$, $n_{\delta,2} = 2$ and $n_{\delta,3} = 4$ shown in Fig. 3. As illustrated, this pattern diagram can be fully decomposed by BIA-feasible 4-tuples.
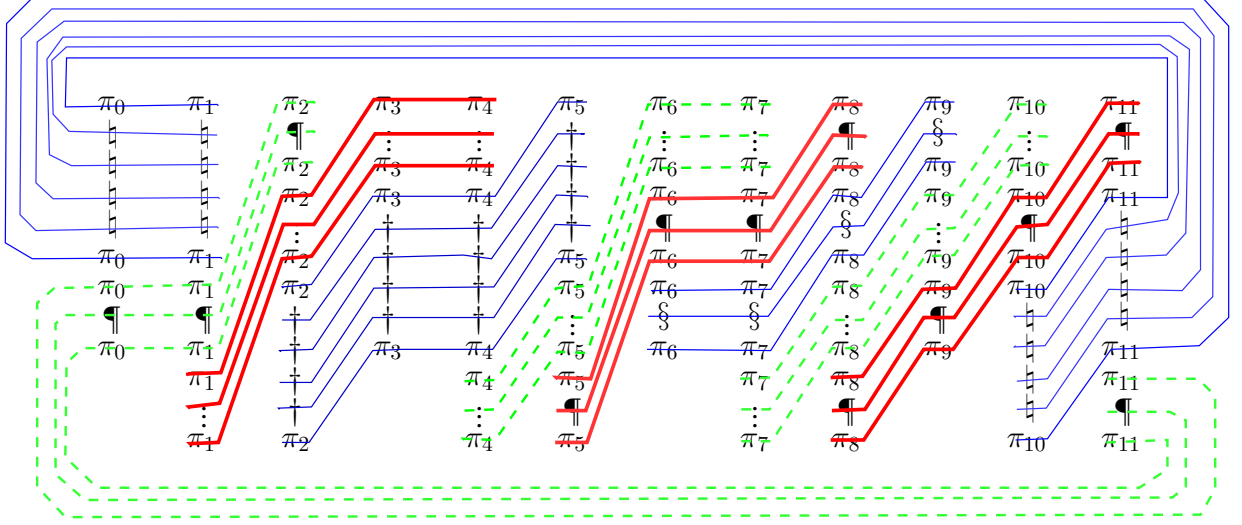

Fig. 6: Complete decomposition of the pattern diagram of a homogeneous 3-user 2×1 BC which meets the sufficient condition given by (8).

It is easy to verify, from Fig. 4 as an example, that any 4-symbol channel block formed by $(\pi_t, \pi_{t+1}, \pi_{t+2}, \pi_{t+3})$ is BIA-feasible, where the subscript of π takes value from $\mathcal{Z}_{12}$, the integer ring on the base of 12. Given a pattern diagram, let us connect the elements π_t by threads, and each thread connects a 4-tuple $(\pi_t, \pi_{t+1}, \pi_{t+2}, \pi_{t+3})$. Then we have the following definition.

Definition 1. We say a pattern diagram is able to be *completely decomposed* into N 4-tuple $(\pi_t, \pi_{t+1}, \pi_{t+2}, \pi_{t+3})$ if each element π_t is connected by one and only one thread.

Fig. 5 shows an example of completely decomposable channel pattern. Then, we have the following lemma.

Lemma 1. A homogeneous 3-user 2×1 BC is BIA-feasible if its pattern diagram is completely decomposable.

By using the lemma, we can find a sufficient condition for a homogeneous 3-user 2×1 to be BIA-feasible in terms of s_i, which is given by the following theorem.

Theorem 1. A homogeneous 3-user 2×1 BC is BIA-feasible if the following inequality is satisfied:

$$\sum_{i=0}^{2} s_i \leq 4\min(s_0, s_1, s_2). \tag{8}$$

Proof: We prove the theorem by showing a constructive algorithm which can completely decomposes a pattern diagram, if it satisfies (8), into N 4-tuple $(\pi_t, \pi_{t+1}, \pi_{t+2}, \pi_{t+3})$.

Without loss of generality, we assume $s_0 = \min(s_0, s_1, s_2)$. Fig. 6 shows the pattern diagram of a homogeneous 3-user 2×1 BC which meets the sufficient condition (8). This figure also shows the threads which are designed to completely decompose the pattern diagram. Let l_t be the number of threads beginning from π_t, $t \in \mathcal{Z}_{12}$. To elaborate the decomposition, we start with the threads beginning with π_1. We use $l_1 = s_1 - s_0$ such threads, and denote $\vdots$ as the π_t's connected by this number of threads. The values of other l_t's and the notations associated with them are listed in Table. I. From $\sum_{i=0}^{2} s_i \leq 4s_0$, we can prove $l_t \geq 0$ for all $t \in \mathcal{Z}_{12}$.

From Table. I, we can easily verify that the equation

$$s_t = l_{t-3} + l_{t-2} + l_{t-1} + l_t \tag{9}$$

	l_0	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}
Size	0	$s_1 - s_0$	$2s_0 - s_1$	0	$s_1 - s_0$	$s_2 - s_0$	$3s_0 - s_1 - s_2$	$s_1 - s_0$	$s_2 - s_0$	0	$2s_0 - s_2$	$s_2 - s_0$
Notations		$\vdots$	†		$\vdots$	¶	§	$\vdots$	¶		♮	¶

TABLE I: The size of the threads beginning from π_t and the notation.

is valid for all $t \in \mathcal{Z}_{12}$. For instance, when $t = 7$, we have $s_7 = s_1$ according to (7), and $l_4 + l_5 + l_6 + l_7 = s_1 - s_0 + s_2 - s_0 + 3s_0 - s_1 - s_2 + s_1 - s_0 = s_1$. This proves that the decomposition given by Table. I, also shown in Fig. 6, is a valid complete decomposition. So, the theorem is proved. ∎

IV. Impact of coherence time N and user number K

Physically, given that $n_{\delta,1} = 0$ is fixed, it is justified to model $n_{\delta,i}$, $i \in \{2,3\}$, as independent random variables uniformly distributed over $0 \le n_{\delta,i} \le N - 1$.

A. Impact of coherence time N

Given N and $n_{\delta,1} = 0$, there are N^2 pairs of $(n_{\delta,2}, n_{\delta,3})$. However, not all of them are able to satisfy the sufficient condition (8) to ensure a BIA-feasible BC. The following lemma counts the number of pairs which meet the sufficient condition.

Lemma 2. Given a homogeneous 3-user 2×1 BC with coherence time N and $n_{\delta,1} = 0$, let $f(N,3)$ denote the number of $(n_{\delta,2}, n_{\delta,3})$ pairs which can generate BIA-feasible channels. Then we have

$$f(N,3) = 2 \sum_{s_0 = \lceil \frac{N}{4} \rceil}^{\lfloor \frac{N}{3} \rfloor} \left(6(\lfloor \tfrac{N-s_0}{2} \rfloor - s_0) + 3 - 3 \left\lfloor \lfloor \tfrac{N-s_0}{2} \rfloor \tfrac{2}{N-s_0} \right\rfloor \right) + 2 \left\lfloor \lfloor \tfrac{N}{3} \rfloor \tfrac{3}{N} \right\rfloor, \tag{10}$$

where $\lfloor \cdot \rfloor$ represents the floor function, $\lceil \cdot \rceil$ represents the ceiling function.

Proof: Let $s_0 \le s_1 \le s_2$. Then according to $\sum_{i=0}^2 s_i = N$ and the sufficient condition $\sum_{i=0}^2 s_i \le 4s_0$, we have the constraint $3s_0 \le N \le 4s_0$. As s_0 must be an integer, we rewrite the constaint on s_0 as $\lceil \tfrac{N}{4} \rceil \le s_0 \le \lfloor \tfrac{N}{3} \rfloor$. Given any s_0 satisfying the constraint, s_1 can take any value from $s_0 \le s_1 \le \lfloor \tfrac{N-s_0}{2} \rfloor$, resulting in $\lfloor \tfrac{N-s_0}{2} \rfloor - s_0 + 1$ choices. Once s_0 and s_1 are chosen, then s_2 is determined by $s_2 = N - s_0 - s_1$.

Suppose $n_{\delta,2} < n_{\delta,3}$. For any triplet (s_0, s_1, s_2), we can assign them to $n_{\delta,2}$ and $n_{\delta,3}$ using $n_{\delta,2} = s_{i1}$, $n_{\delta,3} - n_{\delta,2} = s_{i2}$ and $N - n_{\delta,3} = s_{i3}$, where $(i1, i2, i3)$ is any permutation of $(0,1,2)$. Each triplet gives 6 possible pairs of $(n_{\delta,2}, n_{\delta,3})$ if s_0, s_1 and s_2 are all different; 3 pairs if two of s_i's are equal; 1 pair if all of them are equal. So, given $n_{\delta,2} < n_{\delta,3}$, the number of pairs of $(n_{\delta,2}, n_{\delta,3})$ which meet the sufficient condition is given by

$$g(N,3) = \sum_{s_0 = \lceil \frac{N}{4} \rceil}^{\lfloor \frac{N}{3} \rfloor} \left(6(\lfloor \tfrac{N-s_0}{2} \rfloor - s_0) + 3 - 3 \left\lfloor \lfloor \tfrac{N-s_0}{2} \rfloor \tfrac{2}{N-s_0} \right\rfloor \right) + \left\lfloor \lfloor \tfrac{N}{3} \rfloor \tfrac{3}{N} \right\rfloor, \tag{11}$$

where the constant term '3' in the summation counts the number of pairs when $s_1 = s_0$; the term '$3 \left\lfloor \lfloor \tfrac{N-s_0}{2} \rfloor \tfrac{2}{N-s_0} \right\rfloor$' accounts for the cases when $\tfrac{N-s_0}{2}$ is an integer and thus $s_1 = s_2$; and the term '$\left\lfloor \lfloor \tfrac{N}{3} \rfloor \tfrac{3}{N} \right\rfloor$' accounts for the case when $s_0 = \tfrac{N}{3}$ and $s_2 = s_1 = s_0$.

By realizing the same number of pairs are available if $n_{\delta,2} > n_{\delta,3}$, we get

$$f(N,3) = 2g(N,3), \tag{12}$$

which proves the theorem. ∎

The lemma above clearly shows that the number of BIA-feasible BCs is determined by the coherence time N. Based on the assumption that $n_{\delta,i}$s are independently and uniformly distributed, each pair of $(n_{\delta,2}, n_{\delta,3})$ would occur with the probability of $\tfrac{1}{N^2}$. By using the lemma above, we can derive the probability that a homogeneous 3-user 2×1 BC is BIA-feasible as follows.

Theorem 2. Given a homogeneous 3-user 2×1 BC with coherence time N, the probability that it is BIA-feasible is given by
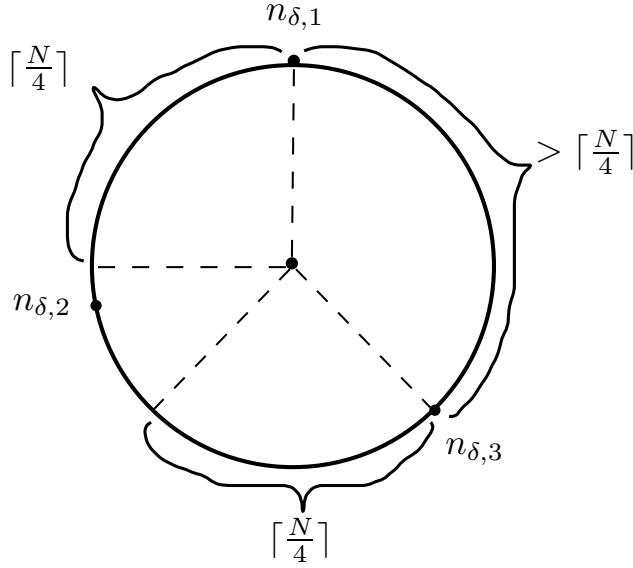
$$P(N,3) = \frac{f(N,3)}{N^2}. \tag{13}$$

Fig. 7: A BIA-feasible 3-tuple $(n_{\delta,1}, n_{\delta,2}, n_{\delta,3})$.

B. Impact of user number K

Now, consider a 2×1 broadcast network with $K \geq 4$ mobile users experiencing homogeneous block fading. Suppose the fading block offset for Rx_i ($2 \leq i \leq K$) is independently and uniformly distributed over $0 \leq n_{\delta,i} \leq N - 1$, with respect to $n_{\delta,1} = 0$. Now, we examine the probability that the 2-antenna transmitter can find, among the K users, three users to form a BIA-feasible 3-user 2×1 BC.

We first give a lemma showing the BIA-feasible condtion on $(n_{\delta,1}, n_{\delta,2}, n_{\delta,3})$.

Lemma 3. Three users form a BIA-feasible BC if $|n_{\delta,i} - n_{\delta,j}| \geq \lceil \frac{N}{4} \rceil$ holds for any pair of $i \neq j$.

Proof: Illustratively, we can visualize the condition by Fig. 7, in which any pair of $n_{\delta,i}$ and $n_{\delta,j}$ is separated by at least $\lceil \frac{N}{4} \rceil$. The lemma can be easily proved by setting one user as the benchmark, say $n_{\delta,1} = 0$, and then applying Theorem 1. ∎

Next we show another lemma which is about to be used in the following analysis.

Lemma 4. Suppose there are n labeled boxes, and Θ labeled balls. Given $\mu \leq \min\{n, \Theta\}$ boxes, the number of ways to put the balls into the boxes such that the μ boxes are not empty is given by

$$\gamma(n, \Theta, \mu) = \sum_{k=\mu}^{\Theta} \binom{\Theta}{k} \mu! S(k, \mu)(n - \mu)^{\Theta-k}, \tag{14}$$

where $S(k, \mu) = \frac{1}{\mu!} \sum_{j=0}^{\mu} (-1)^{\mu-j} \binom{\mu}{j} j^k$ is the Stirling number of the second kind [7].

Proof: We divide the ball assignment process into two steps. Firstly we randomly choose $k \geq \mu$ balls, which has $\binom{\Theta}{k}$ ways, and put the chosen balls into the μ boxes such that each box has at least one balls, which has $\mu! S(k, \mu)$ ways. Secondly we randomly put the rest $\Theta - k$ balls into the rest $n - \mu$ boxes. To combine these two steps and sum over $\mu \leq k \leq \Theta$ proves the lemma. ∎

When there are K users, by using the two lemmas above, we can count the number of $\{n_{\delta,i}\}_{i=2}^{K}$ events in which no three users' offsets can form the feasible ring as shown in Fig. 7. To ease the derivation, we assume N is a multiple of 4, that is, $\frac{N}{4} \in \mathcal{N} = \{1, 2, \cdots\}$.

Theorem 3. Given N the coherence time subject to $\frac{N}{4}$ being an integer, let $f(N, K, 3)$ be the number of $\{n_{\delta,i}\}_{i=2}^{K}$ events in which no three users, from the K users, can help the transmitter to form a BIA-feasible BC. Then

$$
\begin{aligned}
f(N, K, 3) &= 1 + \sum_{n=2}^{N/2} \left(2[n^{K-1} - (n-1)^{k-1}] + (n-2)[n^{K-1} - 2(n-1)^{K-1} + (n-2)^{K-1}] \right) \\
&\quad + (2^{K-1} - 1) + \sum_{n=3}^{N/4+1} (n-1) \left[2(n-3)\gamma(n, K-1, 3) + 3\gamma(n, K-1, 2) \right]
\end{aligned}
$$

(a) Type 1.   (b) Type 2

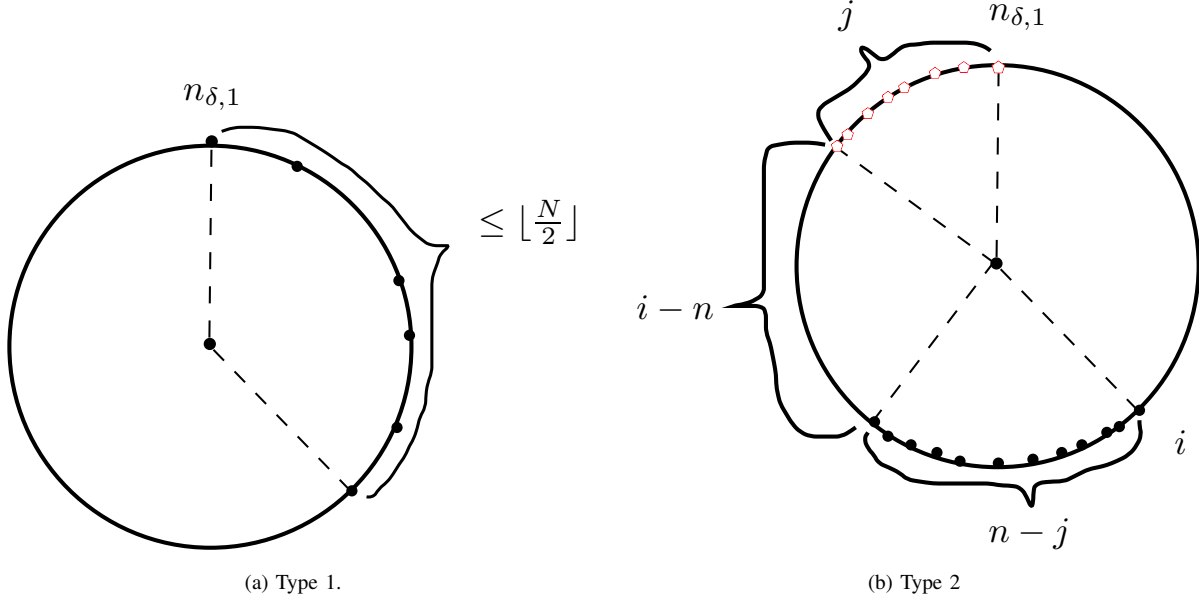Fig. 8: Events with no BIA-feasible 3-tuple.

$$+ \sum_{n=3}^{N/4+1} (n-1)(n-3)\left[\frac{1}{2}(n-4)\gamma(n,K-1,4)+\gamma(n,K-1,3)\right]+$$
$$\sum_{n=N/4+2}^{N/2} (\frac{N}{2}-n+1)(n-1)\left(2\gamma(n,K-1,3)+\frac{1}{2}(n-4)\gamma(n,K-1,4)\right) \tag{15}$$

where

$$\gamma(n,K-1,2) = n^{K-1}-2(n-1)^{K-1}+(n-2)^{K-1}, \tag{16a}$$
$$\gamma(n,K-1,3) = n^{K-1}-3(n-1)^{K-1}+3(n-2)^{K-1}-(n-3)^{K-1}, \tag{16b}$$
$$\gamma(n,K-1,3) = n^{K-1}-4(n-1)^{K-1}+6(n-2)^{K-1}-4(n-3)^{K-1}+(n-4)^{K-1}. \tag{16c}$$

Proof: We cast this problem into a ball-box problem [7], in which N boxes are labeled from 1 to N counter-clockwise forming a ring as shown in Fig. 8, and a user with offset $n_{\delta,k}$ is put into the box with label $n_{\delta,k}+1$. As before, we set $n_{\delta,1}=0$ as the benchmark. We refer to the length of an arc as the number of the boxes in the arc. Then, we divide all events in which no three users can meet the BIA condition into two types. The first is that the length of the arc which contains all users is no greater than $\frac{N}{2}$, and the second larger than $\frac{N}{2}$.

Type I), We start with counting the number of the first type events. Given an arc with the length $n \le \frac{N}{2}$ as shown in Fig. 8a, we count the event number by applying the similar argument developed in [6].

If $n_{\delta,1}$ is one end point of the arc, then the rest $K-1$ users can be randomly loaded into the n boxes on the arc subject to the condition that the other end of the arc must be occupied by at least one user. The number of such events is $2[n^{K-1}-(n-1)^{K-1}]$, where the coefficient 2 reflects $n_{\delta,1}$ can be either of the two end points. If $n_{\delta,1}$ is not any end point of the arc, then the location of the arc relative to $n_{\delta,1}$ has $n-2$ possibilities, and for each possibility the rest $K-1$ users can be randomly loaded into the n boxes subject to that the two end points of the arc must be occupied. The number of such events is $(n-2)[n^{K-1}-2(n-1)^{K-1}+(n-2)^{K-1}]$, i.e., $(n-2)\gamma(n,K-1,2)$. Combining them, and summing over $1 \le n \le \frac{N}{2}$, we get the number of type-I events:

$$f_1(N,K,3)=1+\sum_{n=2}^{N/2}\left(2[n^{K-1}-(n-1)^{K-1}]+(n-2)\gamma(n,K-1,2)\right). \tag{17}$$

Type II), Now we count the type II events. As shown in Fig. 8b, we assume all users are located in two arcs, which occupy n boxes in total; one arc is filled by users denoted by pentagons, the other by users denoted by dots.

(II,a): In this part, we count the events in which $n_{\delta,1}$ is one end point of the pentagon arc. In such events, as shown in Fig. 8b, the position of the end point of the dot arc i should satisfy the condition $\frac{N}{2}+1 \le i \le n+\frac{N}{2}-1$, otherwise the event

would belong to the type I above. Given such a i, it requires that $j \leq \frac{N}{4}$ and $n - j \leq \frac{N}{4}$, otherwise a BIA-feasible 3-tuple will arise according to Lemma 3. In the following we count the number of such events as n takes three different kinds of values.

Given $n = 2$, then $i = \frac{N}{2} + 1$, the rest $K - 1$ balls are randomly put into two boxes subject to that the other box must be occupied, resulting in $2^{K-1} - 1$ such events.

Given $3 \leq n \leq \frac{N}{4} + 1$ and suitable i, the pentagon arc length j can take three kinds of values. Firstly, when $2 \leq j \leq n - 2$, the user $n_{\delta,1}$ can be either of the end points of the arc. So the number of possibilities of the arc combination with $n_{\delta,1}$ being one end point is $2(n-3)$, and the rest $K - 1$ users can be randomly located at the two arcs with totally n boxes subject to that the other three end points of the two arcs must be occupied, resulting in $2(n-3)\gamma(n, K-1, 3)$ events. Secondly, when $j = 1$, the two ends of the pentagon arc is the same. Consequently there are $K - 1$ users to be located subject to that the two end points of the dot arc must be occupied, resulting in $\gamma(n, K-1, 2)$ events. Thirdly, when $j = n - 1$, the dot arc has only one end point to fill, resulting in $2\gamma(n, K-1, 2)$ events, in which 2 reflects $n_{\delta,1}$ can be either of the two end points of the pentagon arc. Combining the cases of $j = 1$, $2 \leq j \leq n - 2$, and $j = n - 1$, we conclude that the number of events for the given n and i is $2(n-3)\gamma(n, K-1, 3) + 3\gamma(n, K-1, 2)$.

Given $\frac{N}{4} + 2 \leq n \leq \frac{N}{2}$ and i, the pentagon arc length can not be $j = 1$ or $j = n - 1$, but $n - \frac{N}{4} \leq j \leq \frac{N}{4}$. Referring to the argument above for $j \neq 1$ and $j \neq n - 1$, we get $2(\frac{N}{2} - n + 1)\gamma(n, K-1, 3)$ such events, where $\frac{N}{2} - n + 1$ stands for the possible choices of j based on $n - \frac{N}{4} \leq j \leq \frac{N}{4}$.

(II.b): Now we consider the events in which $n_{\delta,1}$ is not one end point of the pentagon arc, but an internal point of the arc. It is clear that such events can only happen when $j \geq 3$, and thus $n \geq 4$. We first consider the case of $4 \leq n \leq \frac{N}{4} + 1$. Given such a n and suitable i, if $3 \leq j \leq n - 2$, the user $n_{\delta,1}$ can take $j - 2$ internal positions, resulting in $\sum_{j=3}^{n-2}(j-2)$ possibilities. The rest $K - 1$ users can be randomly located subject to that each of the end points of the two arcs must be occupied by at least one user, resulting in $\sum_{j=3}^{n-2}(j-2)\gamma(n, K-1, 4)$ events. If $j = n - 1$, the dot arc only has one end point, and the three end points must be occupied, generating $(j-2)\gamma(n, K-1, 3) = (n-3)\gamma(n, K-1, 3)$ events. Combining $3 \leq j \leq n - 2$ and $j = n - 1$, we can find $\sum_{j=3}^{n-2}(j-2)\gamma(n, K-1, 4) + (n-3)\gamma(n, K-1, 3)$ such events, for $4 \leq n \leq \frac{N}{4} + 1$ and suitable i.

Then we consider the case of $\frac{N}{4} + 2 \leq n \leq \frac{N}{2}$. Given such n, the length of pentagon arc is constrained by $n - \frac{N}{4} \leq j \leq \frac{N}{4}$, which excludes the chance of $j = n - 1$. Then referring to the argument above for $j \neq n - 1$, we get $\sum_{j=n-N/4}^{N/4}(j-2)\gamma(n, K-1, 4)$ events.

Finally, we note that no matter whichever type-II case above happens, i can take $n + \frac{N}{2} - 1 - (\frac{N}{2} + 1) + 1 = n - 1$ choices (cf. $\frac{N}{2} + 1 \leq i \leq n + \frac{N}{2} - 1$ for $n_{\delta,1}$ as shown in Fig. 8b). In summary, combining all cases above for type II, and summing over the $n - 1$ choices of i, and over either $2 \leq n \leq \frac{N}{4} + 1$ or $\frac{N}{4} + 2 \leq n \leq \frac{N}{2}$, we get the number of type-II events

$$
\begin{aligned}
f_2(N, K, 3) = {} & (2^{K-1} - 1) + \sum_{n=3}^{N/4+1} \sum_{i=N/2+1}^{n+N/2-1} [2(n-3)\gamma(n, K-1, 3) + 3\gamma(n, K-1, 2)] \\
& + \sum_{n=4}^{N/4+1} \sum_{i=N/2+1}^{n+N/2-1} \left[\sum_{j=3}^{n-2}(j-2)\gamma(n, K-1, 4) + (n-3)\gamma(n, K-1, 3) \right] + \\
& \sum_{n=N/4+2}^{N/2} \sum_{i=N/2+1}^{n+N/2-1} \left(2(\frac{N}{2} - n + 1)\gamma(n, K-1, 3) + \sum_{j=n-N/4}^{N/4}(j-2)\gamma(n, K-1, 4) \right)
\end{aligned}
\tag{18}
$$

In summary, the number of events which contains no BIA-feasible 3-tuple is obtained by adding $f_1(N, K, 3)$ and $f_2(N, K, 3)$, which proves the theorem. ∎

By using the theorem above, we can prove the following theorem.

Theorem 4. Given a homogeneous 2×1 broadcast network with K users and N the coherence time subject to $\frac{N}{4}$ being an integer, let $P(N, K, 3)$ be the probability that the transmitter finds, among the K users, three users to form a BIA-feasible 3-user 2×1 BC. Then

$$
P(N, K, 3) = 1 - \frac{f(N, K, 3)}{N^{K-1}}.
\tag{19}
$$

Proof: Based on the fact that the number of possible $\{n_{\delta,i}\}_{i=2}^{K}$ events is N^{K-1} and $n_{\delta,i}$ is uniformly distributed for all $2 \leq i \leq K$, the proof of the theorem is quite straightforward. ∎

V. RESULTS AND DISCUSSIONS

Given a homogeneous 3-user 2×1 BC, Fig. 9 shows its BIA-feasible probability $P(N, 3)$ as a function of coherence time N. When $N = 4$, the BC has the maximal BIA-feasible probability $P(4, 3) = 0.375$. Rather than a monotonic function of N, the probability $P(N, 3)$ forms a Cauchy sequence over N and converges to a constant value when N goes large. In specific, the probability limit $P(\infty, 3)$ falls in $(P(137, 3), P(140, 3)) = (0.06, 0.068)$.
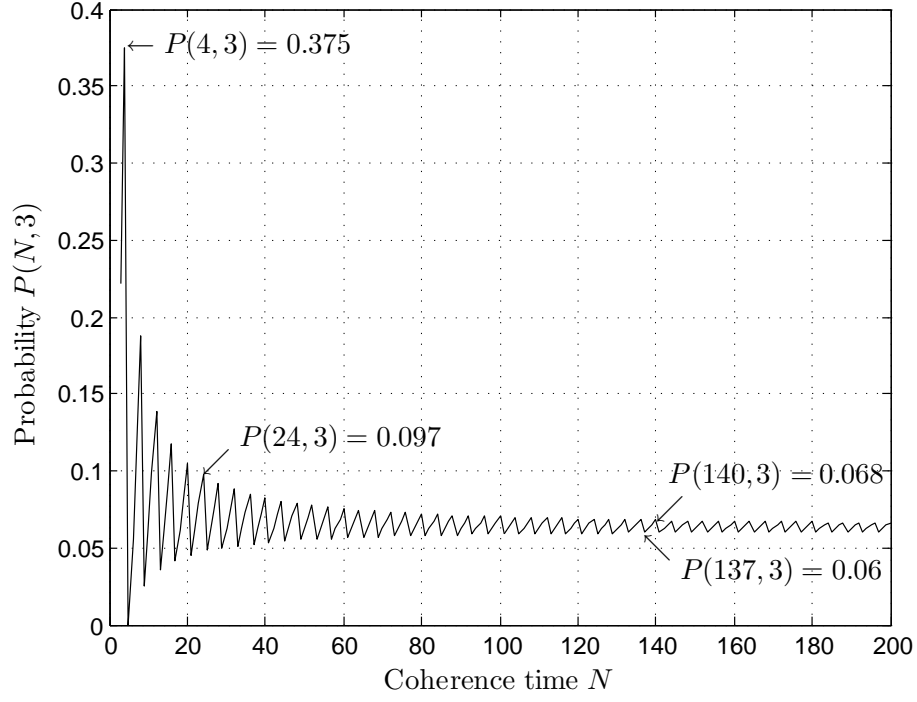
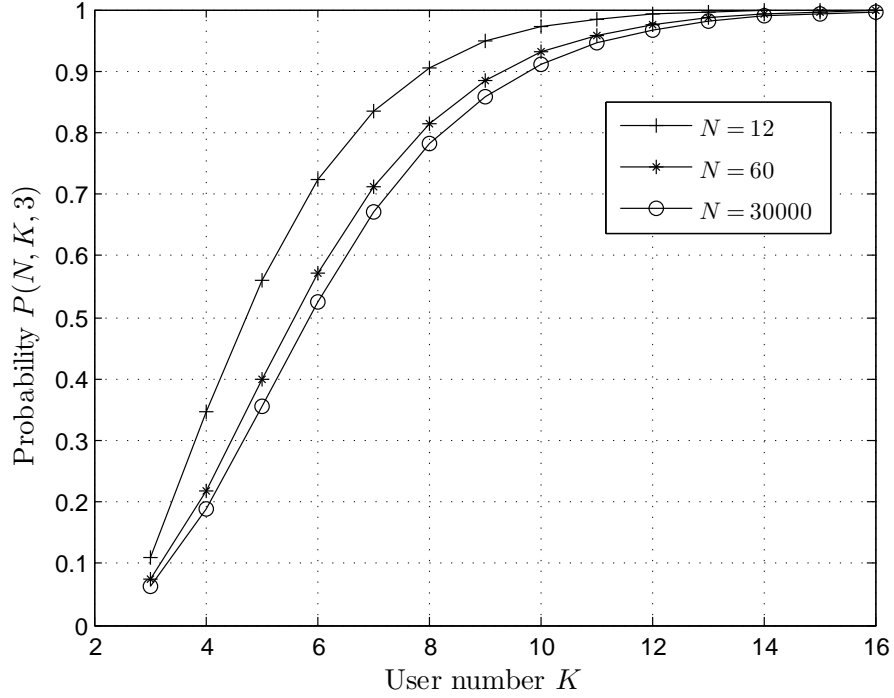Fig. 9: Probability $P(N,3)$ as a function of coherence time N.

Fig. 10: Probability $P(N,K,3)$ as a function of user number K for different coherence time N.

Although a homogeneous 3-user 2×1 BC has a small chance to be BIA-feasible, as presented above, it is expected that the probability increases if the three users can be selected favorably from a number of independent homogeneous users. Quantitatively, Fig. 10 shows the probability $P(N, K, 3)$ as a function of user number K for $N = 12, 60, 3000$. It is observed that, when fixing N, the probability of finding 3 users among K users to form a BIA-feasible BC increases monotonically with K. When $N \geq 60$, the $P(N, K, 3)$ curve is close to the $P(\infty, K, 3)$ curve, which is fairly portrayed by $P(30000, K, 3)$. Based on the $P(30000, K, 3)$ curve, we observe that the probability is larger than 95% for one to find three users from the K users to form a BIA-feasible 3-user BC when $K = 11$. The probability is almost one when $K \geq 15$.

VI. Conclusion

In summary, this paper shows a sufficient condition for a homogeneous 3-user 2×1 BC to be BIA-feasible, that is, to achieve the optimal $\frac{3}{2}$ DoF by using interference alignment without the need of CSIT. Considering a homogeneous 2×1 broadcast network in which the coherence blocks of $K \geq 11$ users are randomly offset with uniform distribution, there is more than 95% chance that the two-antenna transmitter can find three users to form a BIA-feasible 3-user BC. The results in this paper are also applicable to the 2×3 X channel since the 3-user 2×1 BC has the same system model as the 2×3 X channel, in which two single-antenna transmitters broadcast to three single-antenna receivers.

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